

Reply to "A first principles derivation of the electromagnetic fields of a point charge in arbitrary motion"

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In a recently-published article, Singal made a hasty conclusion, "Huang and Lu[1] considered the more general case but found incorrect expressions for the fields".[2] As an author of the article being criticized, I am obliged to respond to such conclusion.

According to the current formulation of the electromagnetic radiation from an accelerated point charge[3, 4, 5, 6], the electric and magnetic fields of the waves radiated from the point charge are (in SI units)

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0 c} \frac{\hat{\mathbf{r}} \times ((\hat{\mathbf{r}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}})}{r(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{r}})^3}, \quad (1)$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0 q}{4\pi} \frac{\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times ((\hat{\mathbf{r}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}))}{r(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{r}})^3}. \quad (2)$$

Here, ϵ_0 and μ_0 are respectively electric permittivity and magnetic permeability of free space, $\boldsymbol{\beta} = \mathbf{v}/c$, \mathbf{v} the velocity of the point charge, q the charge, r the radial distance that the radiated waves travel, and $\hat{\mathbf{r}}$ a unit vector indicating the direction into which the infinitesimal portion of the waves radiate. The electric and magnetic fields of the waves at a position \mathbf{r} and time t are due to the radiation emanating from the charge at the earlier time $t_r = t - r/c$. Thus, the quantities $\boldsymbol{\beta}$ and $\dot{\boldsymbol{\beta}}$ in the right-hand side of Eqs. (1) and (2) are to be evaluated at the retarded time $t_r = t - r/c$. Without using the Liénard-Wiechert potential, rather by the relativistic transformations of velocity, acceleration and electromagnetic fields, Singal obtained the same acceleration field of radiation; the acceleration field of Eq. (37) in the reference [2] is the same as Eq. (1).

However, starting with the electromagnetic fields of radiation from a point charge being instantaneously at rest but accelerated, and using the same relativistic transformations, we obtained the electric and magnetic fields of the waves radiated from a point charge in arbitrary motion [1, 7]

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0 c} \frac{\gamma \hat{\mathbf{r}} \times ((\hat{\mathbf{r}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}})}{r(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{r}})^2}, \quad (3)$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0 q}{4\pi} \frac{\gamma \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times ((\hat{\mathbf{r}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}))}{r(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{r}})^2}, \quad (4)$$

where $\gamma = 1/\sqrt{1 - \beta^2}$. One remarkable distinction between these electromagnetic fields of radiation of the two different derivations is the factor γ . It seems very dubious that Eqs. (1) and (2) lack the γ factor, since all the relativistic transformations of velocity, acceleration and electromagnetic fields contain the γ factor.

We will argue that the currently-accepted expressions Eq. (1) and Eq. (2) are probably incorrect. From these equations, the energy flux density of radiation is

$$\mathbf{S}(\mathbf{r}, t) = \frac{q^2}{16\pi^2\epsilon_0 c} \frac{(\hat{\mathbf{r}} \times ((\hat{\mathbf{r}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}))^2}{r^2(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{r}})^6} \hat{\mathbf{r}}. \quad (5)$$

The energy radiated into a solid angle $d\Omega$ in the direction $\hat{\mathbf{r}}$, and then measured at the position \mathbf{r} and the time t is $dW(\mathbf{r}, t) = \mathbf{S} \cdot \hat{\mathbf{r}} r^2 d\Omega dt$, for an infinitesimal time interval dt . Here $\mathbf{S} \cdot \hat{\mathbf{r}}$ is the energy per unit area per unit time detected at the position \mathbf{r} and the time t of the radiation emitted from the position of the charge at the retarded time $t_r = t - r/c$. Hence, the power radiated per unit solid angle passing through a surrounding sphere

of radius r at the time t is

$$\frac{dP(\hat{\mathbf{r}}, t)}{d\Omega} = \frac{dW(\mathbf{r}, t)}{d\Omega dt} = \frac{q^2}{16\pi^2\epsilon_0 c} \frac{(\hat{\mathbf{r}} \times ((\hat{\mathbf{r}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}))^2}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{r}})^6}. \quad (6)$$

Eq. (6) is the power radiated per unit solid angle measured by observers on the surrounding sphere of radius r . [6] By integrating Eq. (6) over the surrounding sphere, the total radiated power which passes through the sphere at the time t is

$$P(t) = \frac{q^2 a^2 \gamma^8}{6\pi\epsilon_0 c^3} \left(\frac{5 + \beta^2}{5} - \frac{4\beta^2 + 2\beta^4}{5} \sin^2 \alpha \right), \quad (7)$$

where α is the angle between the velocity and the acceleration \mathbf{a} of the charge at the retarded time t_r . Yet, the total power radiated by the charge at the retarded time t_r is given by the Liénard formula, [3, 4, 5, 6]

$$P(t_r) = \frac{q^2 \gamma^6}{6\pi\epsilon_0 c^3} (\mathbf{a}^2 - (\boldsymbol{\beta} \times \mathbf{a})^2) = \frac{q^2 a^2 \gamma^6}{6\pi\epsilon_0 c^3} (1 - \beta^2 \sin^2 \alpha). \quad (8)$$

The total radiated power that passes through any surrounding sphere is $P(t)$. Yet, $P(t)$ is different from the total power radiated by the charge at the retarded time $P(t_r)$. This violates the conservation of energy; the reason is as follows: The total radiated power that passes through a sphere of smaller radius r' ($r' < r$) at an earlier time t' ($t' < t$) is equal to the total radiated power that passes through a sphere of radius r at the time t , since the radiation emitted by the charge expands spherically. As the sphere is getting smaller, and reduced to the position of the charge at the retarded time t_r , $P(t)$ becomes the total power instantaneously radiated by the charge at the retarded time t_r . Therefore, $P(t)$ must be equal to $P(t_r)$ by the conservation of energy. This indicates that the electric and magnetic fields of the waves radiated from an accelerated point charge Eqs. (1) and (2) are incorrect.

In contrast, from Eqs. (3) and (4), the energy flux density of electromagnetic radiation is

$$\mathbf{S}(\mathbf{r}, t) = \frac{q^2}{16\pi^2\epsilon_0 c} \frac{\gamma^2 (\hat{\mathbf{r}} \times ((\hat{\mathbf{r}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}))^2}{r^2 (1 - \boldsymbol{\beta} \cdot \hat{\mathbf{r}})^4} \hat{\mathbf{r}}. \quad (9)$$

The power radiated per unit solid angle passing through any surrounding sphere of radius $r(t)$ at any time t is

$$\frac{dP(\hat{\mathbf{r}}, t)}{d\Omega} = \frac{q^2}{16\pi^2\epsilon_0 c} \frac{\gamma^2 (\hat{\mathbf{r}} \times ((\hat{\mathbf{r}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}))^2}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{r}})^4}. \quad (10)$$

Then, by integrating Eq. (10) over a surrounding sphere, the total radiated power passing through any surrounding sphere is

$$P = \frac{q^2 \gamma^6}{6\pi\epsilon_0 c^3} (\mathbf{a}^2 - (\boldsymbol{\beta} \times \mathbf{a})^2). \quad (11)$$

The total radiated power passing through any surrounding sphere Eq. (11) is equal to the total power radiated by the charge at the retarded time $P(t_r)$; that is, the principle of conservation of energy is satisfied. Therefore, Eqs. (3) and (4) are correct, whereas Eqs. (1) and (2) are not.

We wish that this reply will initiate further examination on the currently-accepted formulation of electromagnetic radiation from an accelerated point charge. Further examination will at least clarify some ambiguities and misunderstanding in the current formulation, if the current formulation is indeed correct.

References

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